Top-dimensional rational cohomology of the congruence subgroup $\Gamma_{0,n}(p)$

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Congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$

Define $\Gamma_{0,n}(p)$ to be

$$\left\{ A \in \mathrm{SL}_n(\mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} * & * & \cdots & \\ 0 & * & & \\ \vdots & & * & \\ 0 & * & & * \end{pmatrix} \bmod p \right\}$$

Theorem

We study the map $\Psi_n\colon \mathrm{H}^{\binom{n}{2}}\left(\Gamma_{0,n}(p);\mathbb{Q}\right) \longrightarrow \widetilde{\mathrm{H}}_{n-2}\left(\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q});\mathbb{Q}\right) \cong \begin{cases} \mathbb{Q} & n=2 \\ 0 & n\geq 3 \end{cases}$

- Ψ_n is always surjective
- Ψ_n is injective if $p \in \{2,3,5,7,13\}$ or $p \leq 6n-14$
- Ψ_n is not injective if $n \in \{2,3\}$ and $p \notin \{2,3,5,7,13\}$

The construction of Ψ_n

- $T_n(\mathbb{Q})$: Tits Building for $\mathrm{GL}_n(\mathbb{Q})$.
- The Solomon-Tits theorem [Sol69] shows that $T_n(\mathbb{Q}) \simeq \bigvee S^{n-2}$.
- Steinberg module $\mathrm{St}_n(\mathbb{Q}) := \widetilde{\mathrm{H}}_{n-2}(T_n(\mathbb{Q});\mathbb{Z})$.
- Borel–Serre [BS73] proved a fundamental result: for any finite index subgroup Γ of $\mathrm{SL}_n(\mathbb{Z})$,
  $$\mathrm{H}^{\binom{n}{2}-k}(\Gamma;\mathbb{Q}) \cong \mathrm{H}_k\left(\Gamma;\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q}\right) \quad \text{for all } k.$$
- Consequently,
  $$\mathrm{H}^k(\Gamma;\mathbb{Q}) \cong 0 \quad \text{for all } k > \binom{n}{2}.$$
- The Borel–Serre duality also implies
  $$\mathrm{H}^{\binom{n}{2}}\left(\Gamma_{0,n}(p);\mathbb{Q}\right) \cong \mathrm{H}_0\left(\Gamma_{0,n}(p);\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q}\right) \cong \left(\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q}\right)_{\Gamma_{0,n}(p)}.$$

The map Ψ_n is constructed in the following way:

- The surjective map
  $$T_n(\mathbb{Q}) \to \Gamma_{0,n}(p)\backslash T_n(\mathbb{Q})$$
  induces a map on homology
  $$\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q} \cong \widetilde{\mathrm{H}}_{n-2}\left(T_n(\mathbb{Q});\mathbb{Q}\right) \to \widetilde{\mathrm{H}}_{n-2}\left(\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q});\mathbb{Q}\right).$$
- Since $\Gamma_{0,n}(p)$ acts trivially on
  $$\widetilde{\mathrm{H}}_{n-2}\left(\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q});\mathbb{Q}\right),$$
  the above map factors through Ψ_n
  $$\left(\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q}\right)_{\Gamma_{0,n}(p)} \to \widetilde{\mathrm{H}}_{n-2}\left(\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q});\mathbb{Q}\right).$$

Quotient of the Tits building $\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q})$

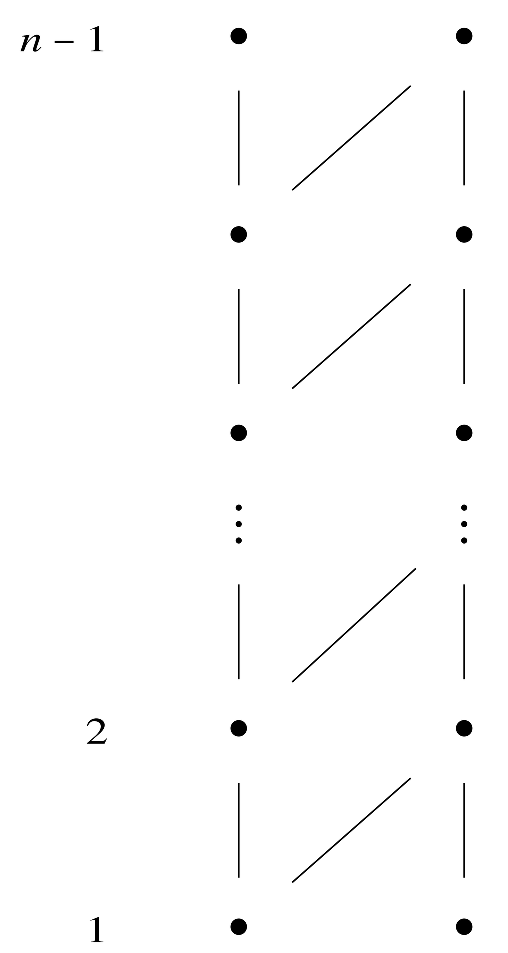

Figure 1. The poset $\mathrm{P}_n(p)$ with realization $|\mathrm{P}_n(p)| = \Gamma_{0,n}(p)\backslash T_n(\mathbb{Q})$.

- As the figure indicates, the quotient $\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q})$ is contractible for $n \geq 3$.
- For $n=2$, the figure shows that $\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q})$ is a discrete two-point set; hence
  $$\widetilde{\mathrm{H}}_0\left(\Gamma_{0,2}(p)\backslash T_2(\mathbb{Q});\mathbb{Q}\right) \cong \mathbb{Q}.$$

Resolution of $\mathrm{St}_n(\mathbb{Q})$

- Complex of partial frames for $\mathbb{Z}^n$, $\mathbf{B}_n(\mathbb{Z})$: a simplicial complex whose k-simplices are partial frames $\{\pm v_0,\ldots,\pm v_k\}$ for $\mathbb{Z}^n$; i.e. $\{v_0,\ldots,v_k\}$ is a partial basis for $\mathbb{Z}^n$.
- Complex of partial augmented frames for $\mathbb{Z}^n$, $\mathbf{BA}_n(\mathbb{Z})$: a simplicial complex whose k-simplices are partial augmented frames $\{\pm v_0,\ldots,\pm v_k\}$ for $\mathbb{Z}^n$; i.e. either $\{v_0,\ldots,v_k\}$ is a partial basis for $\mathbb{Z}^n$ or $\{v_1,\ldots,v_k\}$ is a partial basis for $\mathbb{Z}^n$ with $\pm v_0 \pm v_1 \pm v_2 = 0$ for some choice of signs.

Church–Putman [CP17] and Bykovskiĭ [Byk03] showed that

$$\widetilde{\mathrm{C}}_n(\mathrm{BA}_n(\mathbb{Z});\mathbb{Q}) \to \widetilde{\mathrm{C}}_{n-1}(\mathrm{B}_n(\mathbb{Z});\mathbb{Q}) \to \mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q} \to 0$$

is a flat $\mathrm{SL}_n(\mathbb{Z})$-resolution of $\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q}$.

$\mathrm{B}_n(\mathbb{Z})$ and $\mathrm{BA}_n(\mathbb{Z})$ admit a left $\mathrm{SL}_n(\mathbb{Z})$-action, which implies that the sequence

$$\widetilde{\mathrm{C}}_n(\mathrm{BA}_n(\mathbb{Z});\mathbb{Q})_{\Gamma_{0,n}(p)} \to \widetilde{\mathrm{C}}_{n-1}(\mathrm{B}_n(\mathbb{Z});\mathbb{Q})_{\Gamma_{0,n}(p)} \to (\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q})_{\Gamma_{0,n}(p)} \to 0$$

is exact.

- $\mathbf{BA}_n(\mathbb{Z})'$: subcomplex of $\mathrm{BA}_n(\mathbb{Z})$ consisting of simplices $\{\pm v_0,\ldots,\pm v_k\}$ such that the $\mathbb{Z}$-span of the $\{v_0,\ldots,v_k\}$ is a proper submodule for $\mathbb{Z}^n$.

Then, the exact sequence mentioned above is equivalent to the isomorphism

$$\begin{aligned}&(\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q})_{\Gamma_{0,n}(p)} \\ &\cong \mathrm{H}_{n-1}\left(\mathrm{C}_*(\mathrm{BA}_n(\mathbb{Z}),\mathrm{BA}_n(\mathbb{Z})';\mathbb{Q})_{\Gamma_{0,n}(p)}\right) \\ &\cong \mathrm{H}_{n-1}\left(\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z}),\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z})';\mathbb{Q}\right).\end{aligned}$$

A sketch of the proof of surjectivity of Ψ_n

- Write Ψ_n as
  $$\mathrm{H}_{n-1}\left(\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z}),\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z})';\mathbb{Q}\right) \longrightarrow \widetilde{\mathrm{H}}_{n-2}\left(\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q});\mathbb{Q}\right)$$
- Ψ_n then factors as
  $$\begin{array}{ccc} & \mathrm{H}_{n-1}(\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z}),\ \Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z})';\mathbb{Q}) & \\ \swarrow \partial_n & & \searrow \Psi_n \\ \widetilde{\mathrm{H}}_{n-2}(\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z})';\mathbb{Q}) & \xrightarrow{\Phi_n} & \widetilde{\mathrm{H}}_{n-2}(\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q});\mathbb{Q}) \end{array}$$
  where ∂_n is the boundary map and Φ_n is induced by the poset-map
  $$\mathcal{P}(\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z})') \longrightarrow \mathrm{P}_n(p),$$
  $\mathrm{P}_n(p)$ is the poset whose realization is $\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q})$.
- For $n=2$, we have that
  $$\Gamma_{0,2}(p)\backslash\mathrm{BA}_2(\mathbb{Z})' \cong \Gamma_{0,2}(p)\backslash T_2(\mathbb{Q}).$$
  So, Φ_2 is an isomorphism.
- Moreover, we show
  $$\widetilde{\mathrm{H}}_0\left(\Gamma_{0,2}(p)\backslash\mathrm{BA}_2(\mathbb{Z});\mathbb{Q}\right) \cong 0,$$
  implying that ∂_2 is surjective.
- Together with contractibility of $\Gamma_{0,n}(p)\backslash T_n(\mathbb{Q})$ for $n \geq 3$, we conclude surjectivity.

A sketch of the proof of injectivity of Ψ_n

- Using chain complex and combinatorial arguments, we show that $\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z})$ is highly $\mathbb{Q}$-acyclic whenever $p \in \{2,3,5,7,13\}$ or $p \leq 6n-8$.
- As a consequence, the boundary map ∂_n is injective for the same range of primes.

It remains to show injectivity of Φ_n.

- Using a spectral sequence argument on the poset-map
  $$\mathcal{P}(\Gamma_{0,n}(p)\backslash\mathrm{BA}_n(\mathbb{Z})') \longrightarrow \mathrm{P}_n(p),$$
  we prove that Φ_n is injective whenever $p \in \{2,3,5,7,13\}$ or $p \leq 6n-14$.
- Therefore, Ψ_n is injective for $p \in \{2,3,5,7,13\}$ or $p \leq 6n-14$.

A sketch of the proof of Non-injectivity of Ψ_n

- We give a complete description of the chain groups $\widetilde{\mathrm{C}}_k\left(\Gamma_{0,2}(p)\backslash\mathrm{BA}_2(\mathbb{Z});\mathbb{Q}\right)$.
- We prove that $\Gamma_{0,2}(p)\backslash\mathrm{BA}_2(\mathbb{Z})$ is not highly $\mathbb{Q}$-acyclic if $p \notin \{2,3,5,7,13\}$.
- As a consequence, we obtain the full description of $(\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q})_{\Gamma_{0,n}(p)}$ for $n=2$; which was previously known.

	$p=2,3,5,7,13$	$p=11,17,19$	$p=23,29,31,37$
$(\mathrm{St}_2(\mathbb{Q})\otimes\mathbb{Q})_{\Gamma_{0,2}(p)}$	$\mathbb{Q}$	$\mathbb{Q}^3$	$\mathbb{Q}^5$

Figure 2. $(\mathrm{St}_2(\mathbb{Q})\otimes\mathbb{Q})_{\Gamma_{0,2}(p)}$ for $p \leq 37$.

- Combining this non-acyclicity with our spectral sequence argument, we deduce a new non-vanishing result in the next rank, $n=3$.

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